

# The $n$ -term Approximation of Periodic Generalized Lévy Processes <sup>\*</sup>

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## Abstract

In this paper, we study the compressibility of random processes and fields, called generalized Lévy processes, that are solutions of stochastic differential equations driven by  $d$ -dimensional periodic Lévy white noises. Our results are based on the estimation of the Besov regularity of Lévy white noises and generalized Lévy processes. We show in particular that non-Gaussian generalized Lévy processes are more compressible in a wavelet basis than the corresponding Gaussian processes, in the sense that their  $n$ -term approximation error decays faster. We quantify this compressibility in terms of the Blumenthal-Gettoor index of the underlying Lévy white noise.

## 1 Introduction

Stochastic models are commonly used in engineering and financial applications [10, 21, 23, 25]. The most widely considered stochastic models are Gaussian. However, the Gaussian assumption is too restrictive for many applications, and more general models are needed to accurately represent real data. Stochastic differential equations driven by Lévy noises are able to play this role. We call the solutions of these equations *generalized Lévy processes*. Generalized Lévy processes encompass Gaussian models and the family of compound-Poisson processes that are pure jump processes. They also include symmetric-alpha-stable (S $\alpha$ S) processes which maintain many of the desirable properties of Gaussian models; for example, they satisfy a generalized central-limit theorem [15, 24].

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Generalized Lévy processes with no Gaussian component are particularly relevant to applications [32]. They are called *sparse stochastic processes* [13, 36] due to their ability to model sparse signals [1, 4]. Each member of the family of generalized Lévy processes is associated with a parameter  $\beta \in [0, 2]$ , the Blumenthal-Gettoor index which, as we shall see, determines its sparsity in a wavelet basis. Here, we characterize sparsity by  $n$ -term wavelet approximation rates; in other words, a process is said to be sparser than another one if it satisfies a faster rate of decay of its error of approximation.

Our interest in a sparse wavelet representation is compressibility, since it allows one to store and transmit processes more efficiently by first expanding them in a wavelet basis. We shall show that Gaussian processes (for which  $\beta = 2$ ) are the least compressible, while the compressibility of non-Gaussian processes increases as  $\beta$  decreases. Compressibility is also a useful assumption in inverse imaging problems. If an image is known to be sparse a priori, then this information can be incorporated to produce a better reconstruction algorithm [5, 12, 36].

Our model is the stochastic differential equation

$$Ls = w, \tag{1}$$

where  $L$  is a differential operator,  $s$  is a stochastic process, and  $w$  is a Lévy white noise. The two defining components are the operator and the white noise. The white noise  $w$  is a random periodic generalized function. We consider standard differential-type operators of the following type:  $L$  is assumed to be an order  $\gamma > 0$  operator (typically an order  $\gamma$  derivative) that reduces the regularity of a function by  $\gamma$ .

The compressibility of a given function—and, by extension, of a stochastic process—can be quantified in terms of its Besov regularity. Besov spaces are important function spaces that can be characterized precisely by the rate of decay of their  $n$ -term approximation error [8, 17]. Our strategy to access the compressibility of the considered stochastic process is to connect results on the Besov regularity of the underlying white noise to the compressibility by (deterministic) approximation-theoretic arguments.

The question of the Besov regularity of Lévy processes—that corresponds to (1) in dimension  $d = 1$  with  $L = D$  (the derivative operator)—has been addressed in the literature for subfamilies of Lévy processes [3, 7, 26] as well as in the general case [19]. Several extensions by R. Schilling have been obtained for Lévy-type processes [29, 30]; see also [6, Chapter 5] for a summary. In our case, we aim at considering a multidimensional setting with a general operator  $L$ . This work is therefore a continuation of our previous works on the Besov regularity of Lévy white noises [14, 16].

The rest of the paper is organized as follows: In Section 2, we cover the mathematical background for the remaining sections and discuss function spaces and random processes. In Section 3, we define a class of admissible operators  $L$  for (1), which leads to a definition of generalized Lévy processes on the torus. In Section 4, we put forth the main results of the paper:  $n$ -term approximation of periodic generalized Lévy processes. Finally, we conclude in Section 5 with a discussion of our results and some conjectures to be addressed in a future work.

## 2 Mathematical Background

Motivated by the study of local properties of solutions of stochastic differential equations, this paper deals with periodic random processes. Therefore, we consider spaces of periodic functions in the sequel. We specify these functions on their fundamental domain, the  $d$ -dimensional torus  $\mathbb{T}^d = [-1/2, 1/2)^d \subset \mathbb{R}^d$ ,  $d \geq 1$ .

### 2.1 Lebesgue and Sobolev Spaces

The space of continuous functions on the torus is  $C(\mathbb{T}^d)$ , and  $C^k(\mathbb{T}^d)$  denotes the functions with  $k \in \mathbb{N}$  continuous derivatives. The Lebesgue space  $L_p(\mathbb{T}^d)$  is the collection of functions for which

$$\|f\|_{L_p(\mathbb{T}^d)} := \left( \int_{\mathbb{T}^d} |f(t)|^p dt \right)^{1/p} \quad (2)$$

is finite. For  $p \geq 1$ , (2) is a norm; for  $0 < p < 1$ , it is a quasi-norm. The  $L_2$  Sobolev space of order  $k \in \mathbb{Z}$  is  $H_2^k(\mathbb{T}^d)$ . For  $k \in \mathbb{N}$ ,  $H_2^k(\mathbb{T}^d)$  is the collection of functions in  $L_2(\mathbb{T}^d)$  with  $k$  derivatives in  $L_2(\mathbb{T}^d)$ .

### 2.2 Periodic Test Functions and Generalized Functions

The standard Schwartz space of infinitely differentiable test functions is denoted as  $\mathcal{S}(\mathbb{T}^d)$ . The corresponding space of generalized functions is  $\mathcal{S}'(\mathbb{T}^d)$ . These spaces are nuclear spaces [33], and

$$\mathcal{S}(\mathbb{T}^d) = \bigcap_{k \in \mathbb{Z}} H_2^k(\mathbb{T}^d) \quad (3)$$

$$\mathcal{S}'(\mathbb{T}^d) = \bigcup_{k \in \mathbb{Z}} H_2^k(\mathbb{T}^d). \quad (4)$$

For our purpose, we consider the related spaces with mean zero. Such generalized functions are well suited to wavelet approximation since wavelets also have mean zero. In addition, this assumption simplifies the definition of stochastic processes. For example, the derivative operator becomes a bijective mapping of homogeneous Sobolev and Besov spaces. Moreover, this assumption does not impact the generality of our results: indeed, the addition of a constant term does not affect the regularity of a function.

**Notation 1.** The space of infinitely differentiable test functions with mean zero is denoted as  $\dot{\mathcal{S}}(\mathbb{T}^d)$ . The corresponding space of generalized functions (continuous linear functionals on  $\dot{\mathcal{S}}(\mathbb{T}^d)$ ) is denoted as  $\dot{\mathcal{S}}'(\mathbb{T}^d)$ .

Note that  $\dot{\mathcal{S}}(\mathbb{T}^d)$  can be identified with the quotient of  $\mathcal{S}(\mathbb{T}^d)$  with the one-dimensional space of constant functions on  $\mathbb{T}^d$ . Therefore,  $\dot{\mathcal{S}}(\mathbb{T}^d)$  inherits the nuclear-Fréchet-space structure of  $\mathcal{S}(\mathbb{T}^d)$ . Likewise, we can identify  $\dot{\mathcal{S}}'(\mathbb{T}^d)$  with the space of 0-mean generalized functions  $u \in \mathcal{S}'(\mathbb{T}^d)$ , those for which  $\langle u, 1 \rangle = 0$ .

Also, periodic generalized functions  $f \in \dot{\mathcal{S}}'(\mathbb{T}^d)$  are characterized by their Fourier coefficients  $\langle f, e^{2\pi i \langle \mathbf{m}, \cdot \rangle} \rangle$ ,  $\mathbf{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ . The zero term is excluded because the functions have mean 0.

### 2.3 Generalized Random Processes

**Definition 1.** A *generalized random process* on  $\dot{\mathcal{S}}'(\mathbb{T}^d)$  is a collection of real random variables  $(\langle s, \varphi \rangle)_{\varphi \in \dot{\mathcal{S}}(\mathbb{T}^d)}$  that satisfy the following properties:

- i) *Linearity.*  $\langle s, \varphi + \lambda \psi \rangle = \langle s, \varphi \rangle + \lambda \langle s, \psi \rangle$  almost surely for every  $\varphi, \psi \in \dot{\mathcal{S}}(\mathbb{T}^d)$ .
- ii) *Continuity.*  $\langle s, \varphi_n \rangle \rightarrow \langle s, \varphi \rangle$  in probability whenever  $\varphi_n \rightarrow \varphi$  in  $\dot{\mathcal{S}}(\mathbb{T}^d)$ .

A generalized random process is therefore a continuous and linear functional from  $\dot{\mathcal{S}}(\mathbb{T}^d)$  to the space of random variables. Such an object is called a *continuous linear random functional* in [20].

**Definition 2.** The *characteristic functional*  $\widehat{\mathcal{P}}_s : \dot{\mathcal{S}}(\mathbb{T}^d) \rightarrow \mathbb{C}$  of a generalized random process  $s$  is defined as

$$\widehat{\mathcal{P}}_s(\varphi) = \mathbb{E} \left[ e^{i \langle s, \varphi \rangle} \right]. \quad (5)$$

A generalized random process  $s$  specifies a characteristic functional  $\widehat{\mathcal{P}}_s$  that is positive definite, continuous, and satisfies  $\widehat{\mathcal{P}}_s(0) = 1$ . Conversely, we can define a generalized random process by way of its characteristic functional. This is due to the structure of nuclear Fréchet space  $\dot{\mathcal{S}}'(\mathbb{T}^d)$ .

- The Minlos-Bochner theorem [18] implies that a continuous, positive definite functional  $\widehat{\mathcal{P}} : \dot{\mathcal{S}}(\mathbb{T}^d) \rightarrow \mathbb{C}$  with  $\widehat{\mathcal{P}}(0) = 1$  is the Fourier transform of a probability measure  $\mu$  on  $\dot{\mathcal{S}}'(\mathbb{T}^d)$  endowed with its cylindrical  $\sigma$ -field.
- A probability measure on  $\dot{\mathcal{S}}'(\mathbb{T}^d)$  is uniquely associated to a continuous linear random functional [20, Chapter 2].

We say that a functional  $\widehat{\mathcal{P}}$  is a *valid characteristic functional* if it satisfies the properties of the Minlos-Bochner theorem.

**Example 1.** *The functionals  $\exp\left(-\|\varphi\|_{L_\alpha(\mathbb{T}^d)}^\alpha\right)$  are positive definite and continuous on  $\dot{\mathcal{S}}(\mathbb{T}^d)$  for  $0 < \alpha \leq 2$ . Accordingly, they are valid characteristic functionals that specify some corresponding generalized random processes on  $\dot{\mathcal{S}}'(\mathbb{T}^d)$ , namely,  $S\alpha S$  periodic Lévy white noises.*

## 2.4 Homogeneous Besov Spaces

The homogeneous Besov spaces  $\dot{B}_{p,q}^\tau(\mathbb{T}^d)$  are specified by two primary parameters ( $p$  and  $\tau$ ) and one secondary parameter ( $q$ ). The parameter  $p \in (0, \infty]$  plays a role that is similar to the index defining the Lebesgue spaces  $L_p(\mathbb{T}^d)$ , while  $\tau \in \mathbb{R}$  indicates smoothness in the sense of order of differentiability. Therefore, roughly speaking, for  $\tau \in \mathbb{N}$ , a function in  $\dot{B}_{p,q}^\tau(\mathbb{T}^d)$  has  $\tau$  derivatives in  $L_p(\mathbb{T}^d)$ . In Figure 1, we provide a structural diagram that represents the collection of Besov spaces. Our interest in these spaces lies in the following facts:

- i) Wavelets form unconditional bases for Besov spaces;
- ii) The mapping that takes a function to its wavelet coefficients is an isomorphism between Besov function spaces and Besov sequence spaces;
- iii) The  $n$ -term approximation characterizes Besov sequence spaces. Given a sequence with a known rate of  $n$ -term approximation, we can specify which Besov sequence spaces it is in.

The classical definition of Besov spaces is taken from [34, Definition 1.27] and repeated in Definition 4. The idea is to decompose a function  $f$  by grouping dyadic frequency bands, using a partition of unity in the Fourier domain.

**Definition 3.** Let  $\widehat{v} \in \mathcal{S}(\mathbb{R}^d)$  generate a hierarchical partition of unity outside the ball of radius  $1/2$  centered at the origin. Specifically,

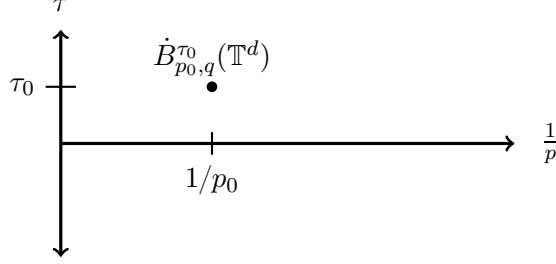


Figure 1: Function-space diagram. A point  $(1/p_0, \tau_0)$  represents all Besov spaces  $\dot{B}_{p_0,q}^{\tau_0}(\mathbb{T}^d)$  for  $0 < q \leq \infty$ .

- i)  $\hat{v}(\omega) = 0$  if  $|\omega| \leq 1/2$  or  $|\omega| \geq 2$
- ii)  $\hat{v}(\omega) > 0$  if  $1/2 < |\omega| < 2$
- iii)  $\sum_{j=0}^{\infty} \hat{v}(2^{-j}\omega) = 1$  if  $1 \leq |\omega|$ .

Note that the decomposition of a function  $f$  into the components

$$\sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \hat{f}(\mathbf{m}) \hat{v}(2^{-j}\mathbf{m}) e^{2\pi i \langle \mathbf{m}, \cdot \rangle} \quad (6)$$

is intimately related to the concept of a wavelet decomposition, with  $v$  playing the role of a mother wavelet.

**Definition 4.** Suppose  $0 < p, q \leq \infty$  and  $\tau \in \mathbb{R}$ . A generalized function  $f \in \mathcal{S}'(\mathbb{T}^d)$  with Fourier coefficients  $\hat{f}(\mathbf{m})$  is in  $\dot{B}_{p,q}^{\tau}(\mathbb{T}^d)$  if the quantity

$$\left( \sum_{j=0}^{\infty} 2^{j\tau q} \left\| \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \hat{f}(\mathbf{m}) \hat{v}(2^{-j}\mathbf{m}) e^{2\pi i \langle \mathbf{m}, \cdot \rangle} \right\|_{L_p(\mathbb{T}^d)}^q \right)^{1/q} \quad (7)$$

is finite. For  $q = \infty$ , the norm must be suitably modified.

Besov spaces are Banach spaces for the norm (7) when  $p$  and  $q \geq 1$ . For  $p$  or  $q < 1$ , (7) is a quasi-norm and Besov spaces are quasi-Banach spaces. The validity of the embeddings between Besov spaces is governed by Proposition 1 [34].

**Proposition 1.** Let  $(-\infty) < \tau_0, \tau_1 < \infty$  and  $0 < p_0, p_1 < \infty$ . Then, the topological embedding  $\dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d) \subseteq \dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)$  is valid in the following cases:

- if  $p_0 \leq p_1$  and  $\tau_0 < \tau_1$ ; or
- if  $p_0 > p_1$  and  $\tau_0 < \tau_1 + d \left( \frac{1}{p_0} - \frac{1}{p_1} \right)$ .

### 3 Generalized Lévy Processes on the Torus

The main objects of study in this paper are generalized Lévy processes  $s$  that are solutions of the stochastic differential equation  $Ls = w$ , with  $w$  a Lévy white noise. If  $w$  has no Gaussian part, then  $s$  is a sparse process. In this section, we introduce the family of Lévy white noises  $w$  and specify the class of considered operators  $L$ .

#### 3.1 Lévy White Noises and Their Besov Regularity

Lévy white noises have been introduced as generalized random processes on the Schwartz space  $\mathcal{D}'(\mathbb{R}^d)$  of generalized functions in [18]. They are commonly defined through their characteristic functional, relying on the Minlos-Bochner theorem (see Section 2.3).

Given a probability space  $\Omega$ , a random variable  $X : \Omega \rightarrow \mathbb{R}$  is infinitely divisible if it can be decomposed as a sum of  $N$  i.i.d. random variables for any  $N \geq 1$  [28]. Lévy white noises are intimately connected with infinite-divisible laws, the finite-dimensional marginals of those processes being all infinitely divisible. The characteristic function of an infinitely divisible random variable  $X$  can be written as

$$\Phi_X(\xi) = \exp(\psi(\xi)) \quad (8)$$

with  $\psi$  a continuous and conditionally positive-definite function with value 0 at  $\xi = 0$  [28, 36]. The continuous log-characteristic function of an infinitely divisible random variable is called a *Lévy exponent* or a *characteristic exponent*. Definition 5 is the adaptation of the usual definition of Lévy white noise [18] to the nuclear space  $\dot{S}'(\mathbb{T}^d)$ .

**Definition 5.** A Lévy exponent  $\psi$  specifies a generalized random process  $w$  in  $\dot{S}'(\mathbb{T}^d)$  with characteristic functional

$$\widehat{\mathcal{P}}_w(\varphi) = \exp \left( \int_{\mathbb{T}^d} \psi(\varphi(\mathbf{x})) d\mathbf{x} \right), \quad (9)$$

where  $\varphi \in \dot{S}(\mathbb{T}^d)$ . We call such a process  $w$  a *Lévy white noise*.

Note that this defines a valid characteristic functional, as seen in Section 2.3 and [18, Chapter 3], where the arguments are easily adapted from  $\mathcal{D}'(\mathbb{R}^d)$  to  $\dot{S}'(\mathbb{T}^d)$ .

The Lévy-Khintchine theorem [28] ensures that a Lévy exponent  $\psi$  can be decomposed as

$$\psi(\xi) = i\mu\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (e^{i\xi t} - 1 - i\xi t 1_{|t| \leq 1}) \nu(dt) \quad (10)$$

where  $\mu \in \mathbb{R}$ ,  $\sigma^2 \geq 0$ , and  $\nu$  is a Lévy measure, which means that it is a measure such that  $\int_{\mathbb{R}} \inf(1, t^2) \nu(dt) < \infty$  and  $\nu\{0\} = 0$ . We say that the white noise is *Gaussian* if  $\mu$  and  $\nu$  are 0. In that case,

$$\widehat{\mathcal{P}}_w(\varphi) = \exp\left(-\frac{\sigma^2 \|\varphi\|_2^2}{2}\right) \quad (11)$$

and we recover the usual Gaussian white noise. If  $\sigma^2 = 0$  (*i.e.*, if  $w$  has no Gaussian part), then we say that  $w$  is *sparse* [36].

We shall deduce the regularity of a solution of (1) from the regularity of the underlying white noise  $w$ . This regularity can be computed in terms of the Blumenthal-Gettoor index of this noise.

**Definition 6.** The *Blumenthal-Gettoor index* (or, simply, the index) of a Lévy white noise with Lévy exponent  $f$  is defined as

$$\beta = \inf \left\{ p \in [0, 2], \limsup_{|\xi| \rightarrow \infty} \frac{|\psi(\xi)|}{|\xi|^p} < \infty \right\}. \quad (12)$$

The Blumenthal-Gettoor index of a Lévy white noise characterizes its local regularity. Theorem 1 was obtained in [14]. It was formulated in terms of local Besov spaces and is easily adapted to the torus. For a direct construction and study of Lévy white noises on the torus, see [16].

**Theorem 1** (Corollary 3, [14]). *We consider a Lévy white noise  $w$  with Blumenthal-Gettoor index  $\beta \in [0, 2]$ . For every  $0 < p, q \leq \infty$ ,  $\tau \in \mathbb{R}$ , if*

$$\tau < d \left( \frac{1}{\max(p, \beta)} - 1 \right), \quad (13)$$

*then  $w \in \dot{B}_{p,q}^\tau(\mathbb{T}^d)$  almost surely.*



### 3.2 Differential Operators of Order $\gamma$

We shall consider the class of differential operators that reduce the Besov regularity of a function by some (possibly fractional) order  $\gamma > 0$ . Importantly, since we are interested in the regularity properties of the solutions of the differential equation  $Ls = w$ , we focus on those operators that are continuous bijections from  $\dot{B}_{p,q}^{\tau+\gamma}(\mathbb{T}^d)$  to  $\dot{B}_{p,q}^\tau(\mathbb{T}^d)$ . In this case, the regularity properties of the white noise allow us to deduce the regularity properties of the process  $s$ .

**Definition 7.** We restrict our attention to the Fourier-multiplier operators  $L : \dot{\mathcal{S}}'(\mathbb{T}^d) \rightarrow \dot{\mathcal{S}}'(\mathbb{T}^d)$ , specified by a symbol  $\hat{L} : \mathbb{R}^d \rightarrow \mathbb{C}$ , where

$$Lf = \mathcal{F}^{-1} \left\{ \hat{f} \hat{L} \Big|_{\mathbb{Z}^d} \right\}. \quad (14)$$

A bijective operator of this form is said to be *admissible*.

**Remark 1.** Let us point out that an admissible operator  $L$  can be initially defined as a mapping from  $\dot{\mathcal{S}}(\mathbb{T}^d)$  to  $\dot{\mathcal{S}}(\mathbb{T}^d)$  since there is a natural extension by duality to the space of generalized functions. Also, notice that we require the symbol  $\hat{L}$  to be defined on the continuous domain  $\mathbb{R}^d$ , even though the action of  $L$  is determined by the values of  $\hat{L}$  on  $\mathbb{Z}^d$ . Our reason will be clear in Theorem 2, where we use the noninteger values on  $\mathbb{R}^d$ .

**Definition 8.** An admissible operator  $L$  is said to be  $\gamma$ -admissible for  $\gamma \in \mathbb{R}$  if  $L : \dot{B}_{p,q}^{\tau+\gamma}(\mathbb{T}^d) \rightarrow \dot{B}_{p,q}^\tau(\mathbb{T}^d)$  is a continuous bijection and  $L^{-1}$  is continuous for every  $0 < p, q \leq \infty$  and  $\tau \in \mathbb{R}$ .

The fractional Laplacian of order  $\gamma > 0$   $(-\Delta)^{\gamma/2}$  is the canonical example of a  $\gamma$ -admissible operator. Moreover, perturbations of the fractional Laplacian are also  $\gamma$ -admissible. The next few results make this statement precise. The idea is the following: An operator  $L$  is  $\gamma$ -admissible if and only if  $(-\Delta)^{\gamma/2}L^{-1}$  and  $(-\Delta)^{-\gamma/2}L$  are automorphisms on Besov spaces.

**Proposition 2.** *The fractional Laplacian  $(-\Delta)^{\gamma/2}$  is a  $\gamma$ -admissible operator.*

*Proof.* This follows from the homogeneity of the symbol of the fractional Laplacian. Applying Theorem 3.3.4 of [31] to Definition 4 gives the result.  $\square$

**Theorem 2.** Let  $L$  be an admissible operator with symbol  $\hat{L}$ . For  $\gamma > 0$ , define  $m(\omega) = |\omega|^{-\gamma} \hat{L}(\omega)$ . Also, let  $\zeta$  be any function in  $\mathcal{S}(\mathbb{R}^d)$  satisfying

$$0 \leq \zeta(\mathbf{x}) \leq 1, \quad \zeta(\mathbf{x}) = \begin{cases} 0, & |\mathbf{x}| \leq 1/4 \\ 1, & 1/2 \leq |\mathbf{x}| \leq 2 \\ 0, & |\mathbf{x}| \geq 4. \end{cases} \quad (15)$$

If

$$\sup_{j \in \mathbb{Z}_{\geq 0}} \left( \|\zeta m(2^j \cdot)\|_{H_2^\tau(\mathbb{R}^d)} + \|\zeta m(2^j \cdot)^{-1}\|_{H_2^\tau(\mathbb{R}^d)} \right) < \infty \quad \text{for all } \tau > 0, \quad (16)$$

then  $L$  is  $\gamma$ -admissible.

*Proof.* This follows from a sufficient condition for Fourier multipliers on Besov spaces, Theorem 3.6.3 of [31]. To summarize, if  $0 < p < \infty$ ,  $0 < q < \infty$ ,  $(-\infty) < \tau < \infty$ , and

$$\tau > d \left( \frac{1}{\min(1, p)} - \frac{1}{2} \right), \quad (17)$$

then there exists a positive constant  $C$  such that

$$\left\| \sum_{\mathbf{k} \in \mathbb{Z}^d} m(\mathbf{k}) \hat{f}(\mathbf{k}) e^{2\pi i \langle \mathbf{k}, \cdot \rangle} \right\|_{\dot{B}_{p,q}^\tau(\mathbb{T}^d)} \leq C \left( \sup_{j \in \mathbb{Z}_{\geq 0}} \|\zeta m(2^j \cdot)\|_{H_2^\tau(\mathbb{R}^d)} \right) \|f\|_{\dot{B}_{p,q}^\tau(\mathbb{T}^d)} \quad (18)$$

holds for all functions  $m \in L_\infty(\mathbb{R}^d)$  and all  $f \in \dot{B}_{p,q}^\tau(\mathbb{T}^d)$ .  $\square$

**Example 2.**  $\gamma$ -Admissible Operators

- i) The derivative  $D$  is 1-admissible.
- ii) The differential operators  $D^\gamma + a_{\gamma-1} D^{\gamma-1} + \dots + a_0 \text{Id}$  with non-vanishing symbols are  $\gamma$ -admissible for  $\gamma \in \mathbb{Z}_{\geq 0}$
- iii) The fractional derivative  $D^\gamma$  is  $\gamma$ -admissible for any  $\gamma > 0$ .
- iv) The fractional Laplacian  $(-\Delta)^{\gamma/2}$  is  $\gamma$ -admissible for any  $\gamma > 0$ .
- v) The Matérn operator  $(\text{Id} - \Delta)^{\gamma/2}$  is  $\gamma$ -admissible for any  $\gamma > 0$ .

### 3.3 Besov Regularity of Generalized Lévy Processes

We are now in a position to deduce the Besov regularity of generalized Lévy processes. Corollary 1 directly follows from the regularity of a white noise process (Theorem 1) and from the definition of a  $\gamma$ -admissible operator.

**Corollary 1.** *We consider a stochastic process  $s = L^{-1}w$ , where  $w$  is a Lévy white noise with Blumenthal-Gettoor index  $\beta \in [0, 2]$  and  $L$  is a  $\gamma$ -admissible operator.*

- *If  $w$  is a Gaussian white noise, then  $s \in \dot{B}_{p,q}^\tau(\mathbb{T}^d)$  almost surely if and only if*

$$\tau < \gamma - \frac{d}{2} \text{ or } \left( \tau = \gamma - \frac{d}{2}, p < \infty, \text{ and } q = \infty \right). \quad (19)$$

- *If  $w$  is a sparse white noise, in the sense that it has no Gaussian part, then  $s \in \dot{B}_{p,q}^\tau(\mathbb{T}^d)$  almost surely if*

$$\tau < \gamma + d \left( \frac{1}{\max(p, \beta)} - 1 \right). \quad (20)$$

*Proof.* By Theorem 1, a sparse white noise is in  $\dot{B}_{p,q}^\tau(\mathbb{T}^d)$  if the parameters satisfy  $\tau < d(1/\max(p, \beta) - 1)$ . For the Gaussian case, we rely on [37], as (19) combines the conditions of Theorem 3.4. The extension of the result from  $w$  to  $s$  is obvious since  $L$  maps bijectively the homogeneous Besov spaces.  $\square$

It is interesting to remark that we have a complete characterization of the Besov localization in the Gaussian case. In particular, a Gaussian process  $s$  is almost surely not in the homogeneous Besov spaces that do not satisfy (19).

## 4 The $n$ -Term Approximation and the Compressibility of Generalized Lévy Processes

In Section 3, we found an upper bound for the Besov regularity of a generalized Lévy process. We are now interested in using that result to determine its  $n$ -term approximation with respect to Daubechies wavelet bases. We begin by defining the wavelets. Then, we recall the wavelet characterization of Besov spaces.

## 4.1 Periodic Daubechies Wavelets

Here, mainly following [34], we introduce the family of Daubechies wavelets on the  $d$ -dimensional torus. We also give a wavelet-based characterization of homogeneous Besov spaces.

Periodizing the compactly supported Daubechies wavelets [9] results in the orthonormal basis of  $L_2(\mathbb{T}^d)$  given by

$$\left\{ \Psi_{G,\mathbf{m}}^{j,k} \mid j \in \mathbb{Z}_{\geq 0}, G \in G^j, \mathbf{m} \in \mathbb{P}_j^d \right\}, \quad (21)$$

where

$$\Psi_{G,\mathbf{m}}^{j,k} = 2^{jd/2} \Psi_{G,0}^{0,k}(2^j \cdot -\mathbf{m}). \quad (22)$$

The index  $j \in \mathbb{Z}_{\geq 0}$  corresponds to a scaling parameter, and  $G$  is used to denote gender. The coarsest scale is  $j = 0$ , which includes the scaling functions, so  $G^0$  has  $2^d$  elements and  $G^j$  has  $(2^d - 1)$  elements for  $j > 0$ . The parameter  $k$  denotes the smoothness of the wavelet and determines its support. For  $k > 0$ , the classical Daubechies mother wavelet on the real line has support greater than one. Here, we require the support of the wavelets to be a subset of the unit cube. Consequently, the coarsest scale is scaled by  $2^L$ , where the parameter  $L \in \mathbb{N}$  ensures that this condition is satisfied. For the remainder of this paper, we set  $L$  (as a function of  $k$ ) to be the smallest integer that guarantees this condition on the support. The wavelet translates are indexed by  $\mathbf{m}$ , and the set of translations at scale  $j$  is

$$\mathbb{P}_j^d = \left\{ \mathbf{m} \in \mathbb{Z}^d \mid 0 \leq m_r < 2^{j+L}, r = 1, \dots, d \right\}. \quad (23)$$

More details on the periodization of wavelet bases can be found in [34, Section 1.3]. In fact, we nearly follow the notation of that book, except that our  $\Psi_{G,\mathbf{m}}^{j,k}$  corresponds to  $\Psi_{G,\mathbf{m}}^{j,\text{per}}$  of [34, Proposition 1.34].

**Definition 9.** The notation  $\Psi_{G_0,0}^{0,k}$  denotes a Daubechies wavelet in  $C^k(\mathbb{T}^d)$ . Furthermore, the Lebesgue measure of the support of  $\Psi_{G_0,0}^{0,k}$  is less than one.

The wavelet decomposition of  $f \in L_2(\mathbb{T}^d)$  is

$$f = \sum_{j,G,\mathbf{m}} \lambda_{\mathbf{m}}^{j,G} 2^{-(j+L)d/2} \Psi_{G,\mathbf{m}}^{j,k}, \quad (24)$$

where the coefficients are computed as

$$\lambda_{\mathbf{m}}^{j,G} = \left\langle f, 2^{(j+L)d/2} \Psi_{G,\mathbf{m}}^{j,k} \right\rangle. \quad (25)$$

We can now state the characterization of the homogeneous Besov spaces  $\dot{B}_{p,q}^\tau(\mathbb{T}^d)$  that will be used in the remainder of this paper. Essentially, it says that the Besov regularity of a function is characterized by a norm on its wavelet coefficients.

**Definition 10** (Definition 1.3.2, [34]). Let  $\tau \in \mathbb{R}$  and  $0 < p, q < \infty$ . The Besov sequence space  $b_{p,q}^\tau$  is the collection of sequences

$$\lambda = \{\lambda_{\mathbf{m}}^{j,G} \mid j \in \mathbb{Z}_{\geq 0}, G \in G^j, \mathbf{m} \in \mathbb{P}_j\} \quad (26)$$

indexed as the periodized Daubechies wavelets, with finite (quasi-)norm

$$\|\lambda\|_{b_{p,q}^\tau} := \left( \sum_{j=0}^{\infty} 2^{j(\tau-d/p)q} \sum_{G \in G^j} \left( \sum_{\mathbf{m} \in \mathbb{P}_j^n} |\lambda_{\mathbf{m}}^{j,G}|^p \right)^{q/p} \right)^{1/q}. \quad (27)$$

If  $p = \infty$  or  $q = \infty$ , there is an analogous definition.

In the special case  $p = q$ , we have

$$\|\lambda\|_{b_{p,p}^\tau} := \left( \sum_{j,G,\mathbf{m}} \left| 2^{j(\tau-d/p)} \lambda_{\mathbf{m}}^{j,G} \right|^p \right)^{1/p}, \quad (28)$$

which is a weighted  $\ell_p$  space.

**Proposition 3.** Suppose  $f \in \dot{B}_{2,2}^{-k}(\mathbb{T}^d)$  (which is equivalent to the Sobolev space  $\dot{H}_2^{-k}(\mathbb{T}^d)$ ) for some  $k \in \mathbb{N}$ , and let  $0 < p, q \leq \infty$  and  $\tau \in \mathbb{R}$  such that  $k > \max(\tau, \sigma_p - \tau)$ , where  $\sigma_p = d(1/p - 1)_+$ . Then,  $f \in \dot{B}_{p,q}^\tau(\mathbb{T}^d)$  if and only if the wavelet coefficients

$$\left\langle f, 2^{(j+L)d/2} \Psi_{G,\mathbf{m}}^{j,k+1} \right\rangle \quad (29)$$

are in the Besov sequence space  $b_{p,q}^\tau$ .

*Proof.* This follows from Theorem 1.36 of [31]. □

## 4.2 Besov Spaces and $n$ -Term Approximations

Summarizing the results of Section 4.1, we have the following: If  $f$  is in the Besov space  $B_{p,q}^\tau(\mathbb{T}^d)$ , then we can choose  $k$  large enough so that

$$f = \sum_{j,G,\mathbf{m}} \lambda_{\mathbf{m}}^{j,G} 2^{-(j+L)d/2} \Psi_{G,\mathbf{m}}^{j,k}, \quad (30)$$

where the coefficients  $\lambda_{\mathbf{m}}^{j,G}$  are computed by (25). We are now going to determine the error in approximating  $f$  by truncating the sum (30). In order to accomplish this, we introduce the notation  $\mathcal{I} = \mathbb{Z}_{\geq 0} \times G^j \times \mathbb{P}_j^d$  to represent the collection of triples  $(j, G, \mathbf{m})$ .

**Definition 11.** For  $n \geq 1$ , an  $n$ -term approximation to a function  $f$  is a finite sum of the form

$$\sum_{(j,G,\mathbf{m}) \in \mathcal{I}'} \lambda_{\mathbf{m}}^{j,G} 2^{-(j+L)d/2} \Psi_{G,\mathbf{m}}^{j,k}, \quad (31)$$

where  $\mathcal{I}' \subset \mathcal{I}$  and  $\#\mathcal{I}' = n$ .

**Definition 12.** Let  $\Sigma_{n,p,\tau}(f)$  be a best  $n$ -term approximation to  $f$  in  $\dot{B}_{p,p}^\tau(\mathbb{T}^d)$ , in other words, an  $n$ -term approximation that minimizes the approximation error in  $\dot{B}_{p,p}^\tau(\mathbb{T}^d)$ . Also, let  $\sigma_{n,p,\tau}(f)$  denote the error of the approximation, with

$$\sigma_{n,p,\tau}(f) = \|f - \Sigma_{n,p,\tau}(f)\|_{\dot{B}_{p,p}^\tau(\mathbb{T}^d)}. \quad (32)$$

**Theorem 3.** Suppose  $0 < p_0 < \infty$  and  $\tau_0 \in \mathbb{R}$ .

i) If  $f \in \dot{B}_{p_1,p_1}^{\tau_0+\Delta\tau}(\mathbb{T}^d)$  for some  $\Delta\tau > 0$  and  $p_1$  satisfies (35), then there is a constant  $C > 0$  such that

$$\sigma_{n,p_0,\tau_0}(f) \leq C n^{-\Delta\tau/d} \|f\|_{\dot{B}_{p_1,p_1}^{\tau_0+\Delta\tau}(\mathbb{T}^d)}. \quad (33)$$

ii) If there are constants  $C, \Delta\tau, \epsilon > 0$  such that

$$\sigma_{n,p_0,\tau_0}(f) \leq C n^{-\Delta\tau/d-\epsilon}, \quad (34)$$

then  $f \in \dot{B}_{p_1,p_1}^{\tau_0+\Delta\tau}(\mathbb{T}^d)$ , where

$$\frac{1}{p_1} = \frac{\Delta\tau}{d} + \frac{1}{p_0}. \quad (35)$$

*Proof.* This proof uses Corollary 6.2 of [17], which characterizes  $n$ -term approximation spaces as Besov spaces. In particular,

$$b_{p_1,p_1}^{\tau_0+\Delta\tau} = A_{p_1}^{\Delta\tau/d}(b_{p_0,p_0}^{\tau_0}), \quad (36)$$

where  $b_{p_1,p_1}^{\tau_0+\Delta\tau}$  is a Besov sequence space, and  $A_{p_1}^{\Delta\tau/d}(b_{p_0,p_0}^{\tau_0})$  is an approximation space with error measured in  $b_{p_0,p_0}^{\tau_0}$ . Essentially,  $A_{p_1}^{\Delta\tau/d}(b_{p_0,p_0}^{\tau_0})$  is the collection of sequences  $f$  for which the sequence of error terms

$$n^{\Delta\tau/d} \sigma_{n,p_0,\tau_0}(f) \quad (37)$$

is in  $\ell_{p_1}$  with respect to a Haar-type measure on  $\mathbb{N}$ .

Using this characterization along with standard embedding properties of approximation spaces [11, Chapter 7], we derive our result. In particular, (36) together with the aforementioned embedding implies that

$$b_{p_1, p_1}^{\tau_0 + \Delta\tau} \subset A_\infty^{\Delta\tau/d}(b_{p_0, p_0}^{\tau_0}). \quad (38)$$

Similarly, we have that

$$A_\infty^{\Delta\tau/d+\epsilon}(b_{p_0, p_0}^{\tau_0}) \subset b_{p_1, p_1}^{\tau_0 + \Delta\tau}. \quad (39)$$

The fact that the continuous-domain Besov spaces are isomorphic to Besov sequence spaces (Proposition 3) completes the proof.  $\square$

### 4.3 The Compressibility of Generalized Lévy Processes

The compressibility of a function quantifies the speed of convergence of its  $n$ -term approximation in a wavelet basis.

**Definition 13.** For a generalized function  $f \in \dot{B}_{p_0, p_0}^{\tau_0}(\mathbb{T}^d)$ , we define its  $(p_0, \tau_0)$ -compressibility as

$$\kappa_{p_0, \tau_0}(f) = \sup \left\{ \lambda \geq 0 \left| \sup_{n \in \mathbb{Z}_{\geq 0}} (n+1)^\lambda \|f - \Sigma_{n, p_0, \tau_0}(f)\|_{\dot{B}_{p_0, p_0}^{\tau_0}(\mathbb{T}^d)} < \infty \right. \right\} \in [0, \infty]. \quad (40)$$

The quantity  $\kappa_{p_0, \tau_0}(f)$  is well-defined and nonnegative for  $f \in \dot{B}_{p_0, p_0}^{\tau_0}(\mathbb{T}^d)$ . The higher  $\kappa_{p_0, \tau_0}(f)$  is, the more  $f$  is compressible. If the approximation error has a faster-than-algebraic decay, then  $\kappa_{\tau_0, p_0}(f) = +\infty$ .

**Remark 2.** One can interpret compressibility in terms of approximation spaces. The sequence  $E_n(f) := \|f - \Sigma_{n, p_0, \tau_0}(f)\|_{\dot{B}_{p_0, p_0}^{\tau_0}(\mathbb{T}^d)}$  specifies how well  $f$  can be approximated by wavelets. Weighted norms of this sequence are commonly used to characterize approximation properties, and such norms specify approximation spaces  $A_q^\lambda$ . For example

$$\|f\|_{A_q^\lambda} := \begin{cases} \left( \sum_{n=1}^{\infty} (n^\lambda E_{n-1}(f))^q \frac{1}{n} \right)^{1/q}, & 0 < q < \infty \\ \sup (n^\lambda E_{n-1}(f)), & q = \infty. \end{cases} \quad (41)$$

Therefore,  $\kappa_{p_0, \tau_0}(f)$  is the largest  $\lambda$  such that  $f \in A_\infty^\lambda$ .

We first give a general result on the  $n$ -term approximation of a sparse process  $s = L^{-1}w$  in  $\dot{B}_{p_0, p_0}^{\tau_0}(\mathbb{T}^d)$ .

**Proposition 4.** *We consider a stochastic process  $s = L^{-1}w$  with Blumenthal-Gettoor index  $\beta \in [0, 2]$  and order  $\gamma \geq 0$ . If  $\tau_0 \in \mathbb{R}$  and  $0 < p_0 \leq \infty$  satisfy the relation*

$$\gamma > \tau_0 + d - \frac{d}{\max(p_0, \beta)}, \quad (42)$$

*then we have the order relation*

$$\sigma_{n,p_0,\tau_0}(s) \leq C n^{\gamma/d+1/\beta-1} \|s\|_{\dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)} \quad (43)$$

*almost surely, where  $\tau_1$  and  $p_1$  are set as*

$$\tau_1 - \tau_0 = \gamma + \frac{d}{\beta} - d = \frac{d}{p_1} - \frac{d}{p_0} \quad (44)$$

*and  $C$  is a (deterministic) constant.*

*Proof.* Due to Corollary 1, Condition (42) ensures that  $s \in \dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)$ . If  $s \notin \dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)$ , then  $\|s\|_{\dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)} = \infty$  so that (43) is obvious. Otherwise, we apply Part i) of Theorem 3 with  $\Delta\tau = (\tau_1 - \tau_0) = (\gamma/d + 1/\beta - 1)$  to deduce the result.  $\square$

In particular, we deduce from Proposition 4 that, under Condition (42), one has almost surely that

$$\sigma_{n,p_0,\tau_0}(s) \xrightarrow[n \rightarrow \infty]{} 0. \quad (45)$$

Based on our preliminary work, we now obtain new results on the compressibility of processes  $s$  that are solutions of (1). We split the results into two cases: the Gaussian processes and the sparse processes.

**Theorem 4** (Compressibility of Gaussian processes). *Let  $s = L^{-1}w_G$  be a Gaussian process of order  $\gamma$ . For  $0 < p_0 \leq \infty$  and if  $\tau_0 \in \mathbb{R}$  satisfies*

$$\gamma > \tau_0 + \frac{d}{2}, \quad (46)$$

*then we have, almost surely, that*

$$\kappa_{p_0,\tau_0}(s) = \frac{\gamma - \tau_0}{d} - \frac{1}{2}. \quad (47)$$

*Proof.* Fix  $0 < \epsilon < (\gamma - d/2 - \tau_0)$ . Then, according to Corollary 1,  $s \in \dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)$  with

$$\tau_1 = \gamma - \frac{d}{2} - \epsilon \text{ and } \tau_1 - \tau_0 = d \left( \frac{1}{p_1} - \frac{1}{p_0} \right). \quad (48)$$



Then, Proposition 4 implies that

$$\sigma_{n,p_0,\tau_0}(s) \leq C n^{\frac{1}{p_1} - \frac{1}{p_0}} \|s\|_{\dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)} \quad (49)$$

with  $C > 0$  a constant. This shows that

$$\kappa_{p_0,\tau_0} \geq \frac{1}{p_1} - \frac{1}{p_0} = \frac{\gamma - \tau_0}{d} - \frac{1}{2} - \frac{\epsilon}{d}. \quad (50)$$

This is valid for  $\epsilon$  arbitrarily small, implying that  $\kappa_{p_0,\tau_0}(s) \geq ((\gamma - \tau_0)/d - 1/2)$ .

We know that  $s$  is not in  $\dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)$  for  $\tau_1 \geq (\gamma - d/2)$  and  $p_1$  given by  $(\tau_1 - \tau_0) = d(1/p_1 - 1/p_0)$ . The converse of Theorem 3, Part ii), implies that there exists no constant  $C > 0$  such that  $\sigma_{n,p_0,\tau_0}(s) \leq C n^{-\gamma-d/2}$ . Therefore,  $\kappa_{p_0,\tau_0}(s) \leq ((\gamma - \tau_0)/d - 1/2)$ , which finishes the proof.  $\square$

**Theorem 5** (Compressibility of sparse processes). *Let  $s = L^{-1}w$  be a sparse process of order  $\gamma$  and Blumenthal-Gettoor index  $\beta$ . We assume that  $\tau_0 \in \mathbb{R}$  and  $0 < p_0 \leq \infty$  satisfy*

$$\gamma > \tau_0 + d - \frac{d}{p_0}. \quad (51)$$

- If  $\beta = 0$ , then, almost surely,

$$\kappa_{p_0,\tau_0}(s) = +\infty. \quad (52)$$

- If  $\beta > 0$ , then, almost surely,

$$\kappa_{p_0,\tau_0}(s) \geq \frac{\gamma - \tau_0}{d} + \frac{1}{\beta} - 1. \quad (53)$$

*Proof.* We first assume that  $\beta > 0$ . We proceed as in the Gaussian case. Condition (51) allows us to consider  $0 < \epsilon < (\gamma - \tau_0) + d(1/p_0 - 1)$  (and also implies that  $s \in \dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)$ , due to Corollary 1). We set  $\tau_1, p_1$  such that

$$\tau_1 = \gamma + \frac{d}{\beta} - d - \epsilon \quad \text{and} \quad \frac{d}{p_1} - \frac{d}{p_0} = \gamma + \frac{d}{\beta} - d - \tau_0 - \epsilon. \quad (54)$$

We claim that  $s \in \dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)$ . Based on Corollary 1, this is true if

$$\tau_1 < \gamma + \frac{d}{\max(p_1, \beta)} - d. \quad (55)$$

Then, we first remark that

$$\frac{1}{p_1} = \frac{1}{\beta} + \left( \gamma - \tau_0 + \frac{d}{p_0} - d - \epsilon \right) > \frac{1}{\beta}, \quad (56)$$

so that  $p_1 \leq \beta$ . Moreover,

$$\tau_1 = \gamma + \frac{d}{\beta} - d - \epsilon \quad (57)$$

$$< \gamma + \frac{d}{\beta} - d \quad (58)$$

$$= \gamma + \frac{d}{\max\{p_1, \beta\}} - d, \quad (59)$$

as expected. Due to Theorem 3, Part i), one deduces that

$$\kappa_{p_0, \tau_0}(s) \geq \frac{\gamma - \tau_0}{d} + \frac{1}{\beta} - 1 - \frac{\epsilon}{d} \quad (60)$$

for  $\epsilon$  arbitrarily small and, therefore, (53) is shown.

For  $\beta = 0$ , we know that  $s \in \dot{B}_{p,p}^\tau(\mathbb{T}^d)$  when  $\tau < (\gamma + d/p - d)$  (Corollary 1). For  $p_1 < p_0$ , we define  $\tau_1 = (\tau_0 + d/p_1 - d/p_0)$ . Then, using (51), we have that

$$\tau_1 = \left( \tau_0 - \frac{d}{p_0} \right) + \frac{d}{p_1} < \gamma + \frac{d}{p_1} - d \quad (61)$$

and, therefore,  $s \in \dot{B}_{p_1, p_1}^{\tau_1}(\mathbb{T}^d)$ . With Theorem 3, we deduce that  $\kappa_{p_0, \tau_0}(s) \geq (1/p_1 - 1/p_0)$  for any  $p_1$  arbitrarily small. Finally, this means that  $\kappa_{p_0, \tau_0}(s) = +\infty$  almost surely.  $\square$

Note that Theorem 4 gives the exact value of  $\kappa_{p_0, \tau_0}(s)$  in the Gaussian case, while we only provide a lower bound for sparse processes in Theorem 5 (except when  $\beta = 0$ ).

## 5 Discussion and Examples

In this section, we focus on the  $L_2$ -compressibility that is obtained for  $p_0 = 2$  and  $\tau_0 = 0$ . This case is of special interest, in particular for signal-processing applications: The quantity  $\kappa(f) = \kappa_{0,2}(f)$  measures the approximation decay in the  $L_2$ -sense. We therefore reformulate Theorems 4 and 5 for this case.

**Corollary 2** ( $L_2$ -compressibility of Gaussian and sparse processes). *Let  $s = L^{-1}w$  be a process of order  $\gamma > d/2$  and of Blumenthal-Gettoor index  $\beta \in [0, 2]$ .*

- *If  $w = w_G$  is a Gaussian white noise, then, almost surely,*

$$\kappa(s) = \frac{\gamma}{d} - \frac{1}{2}. \quad (62)$$

- If  $w$  is sparse with  $\beta = 0$ , then, almost surely,

$$\kappa(s) = +\infty \quad (63)$$

- If  $w$  is sparse with  $\beta > 0$ , then, almost surely,

$$\kappa(s) \geq \frac{\gamma}{d} + \frac{1}{\beta} - 1. \quad (64)$$

We introduce some classical families of Lévy white noises with their indices in Table 1. Precise definitions can be found in the proposed references. The compressibility of the associated  $\gamma$ -admissible processes is deduced from Corollary 2. We use the lower bound of (64) when the exact value  $\kappa(s)$  is not known.

Table 1: Compressibility of Gaussian and sparse processes of order  $\gamma > d/2$  based on specific Lévy white noises

White noise $w$	Parameter	$\psi(\xi)$	$\beta$	Compressibility
Gaussian	$\sigma^2 > 0$	$-\sigma^2 \xi^2/2$	2	$\gamma - \frac{d}{2}$
Cauchy [27]	—	$- \xi $	1	$\gamma$
S $\alpha$ S [27]	$\alpha \in (0, 2)$	$- \xi ^\alpha$	$\alpha$	$\gamma + d/\alpha - d$
Compound Poisson [35]	$\lambda > 0, \mathbb{P}_J$	$\exp(\lambda(\widehat{\mathbb{P}}_J(\xi) - 1))$	0	$\infty$
Laplace [22]	—	$-\log(1 + \xi^2)$	0	$\infty$
Inverse Gaussian [2]	—	—	1/2	$\gamma + d$

We finish with some important remarks and conjectures.

- When  $\beta > 0$ , we have only obtained a lower bound for the compressibility of a generalized Lévy process. We conjecture that our bound actually gives the exact rate of compressibility  $\kappa(s)$ .
- In the Gaussian case, one has the exact value of  $\kappa(s)$ . This is due to the fact that the converse result for the Besov regularity of Gaussian white noises are known [37]. A fundamental consequence is the following: If  $s = L^{-1}w$  is a sparse process of order  $\gamma$  and  $s_G = L^{-1}w_G$  is a Gaussian process corresponding to the same operator  $L$ , then we have almost surely that

$$\kappa(s) \geq \kappa(s_G). \quad (65)$$

Simply stated, sparse processes are more compressible than Gaussian processes. This finally gives a functional justification for the terminology of *sparse* processes introduced in [36].

- For fixed  $\gamma$ , the smaller  $\beta$  is, the more compressible is the process. At the limit, for  $\beta = 0$ , the error decays faster than any polynomial. This is achieved by compound-Poisson processes and also by Laplace processes.
- For fixed  $\beta > 0$ , the compressibility of the process increases with  $\gamma$ . This is a very reasonable outcome as it is well known that smoother functions are more compressible.

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